CS 188: Artificial Intelligence
Spring 2010

Lecture 4: A* wrap-up + Constraint Satisfaction

1/28/2010

Pieter Abbeel – UC Berkeley

Many slides from Dan Klein

Announcements

- Project 0 (Python tutorial) is due today
  - If you don't have a class account yet, pick one up after lecture
- Written 1 (Search) is due today
- Project 1 (Search) is out and due next week Thursday
- Section/Lecture

Recap: Search

- Search problem:
  - States (configurations of the world)
  - Successor function: a function from states to lists of (state, action, cost) triples; drawn as a graph
  - Start state and goal test

General Tree Search

```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
   initialize the search tree using the initial state of problem
   loop do
      if there are no candidates for expansion then return failure
      choose a leaf node for expansion according to strategy
      if the node contains a goal state then return the corresponding solution
      else expand the node and add the resulting nodes to the search tree
   end
```

- Important ideas:
  - Fringe
  - Expansion
  - Exploration strategy

Detailed pseudocode is in the book!

- Main question: which fringe nodes to explore?

A* Review

- A* uses both backward costs g and forward estimate h: f(n) = g(n) + h(n)
- A* tree search is optimal with admissible heuristics (optimistic future cost estimates)
- Heuristic design is key: relaxed problems can help

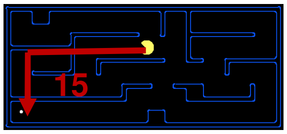
Admissible Heuristics

- A heuristic h is ***admissible*** (optimistic) if:

$$h(n) \leq h^*(n)$$

  where $h^*(n)$ is the true cost to a nearest goal

- Example:

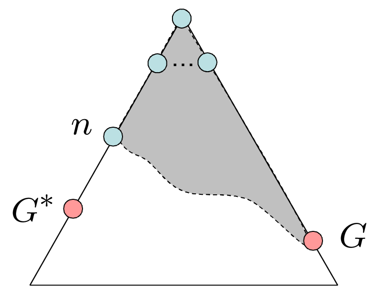


- Coming up with admissible heuristics is most of what's involved in using A* in practice.

Optimality of A*: Blocking

Notation:

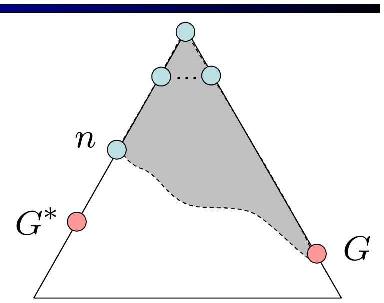
- g(n) = cost to node n
- h(n) = estimated cost from n to the nearest goal (heuristic)
- f(n) = g(n) + h(n) = estimated total cost via n
- G*: a lowest cost goal node
- G: another goal node



Optimality of A*: Blocking

Proof:

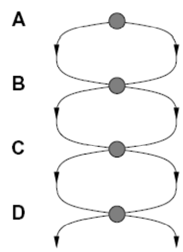
- What could go wrong?
- We'd have to have to pop a suboptimal goal G off the fringe before G*
- This can't happen:
  - Imagine a suboptimal goal G is on the queue
  - Some node *n* which is a subpath of G* must also be on the fringe (why?)
  - *n* will be popped before G

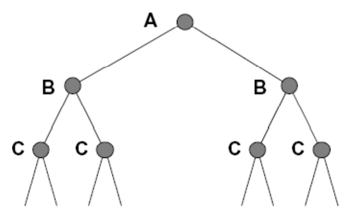


$$f(n) = g(n) + h(n)$$
$$g(n) + h(n) \leq g(G^*)$$
$$g(G^*) < g(G)$$
$$g(G) = f(G)$$
$$f(n) < f(G)$$

Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work. Why?





Graph Search

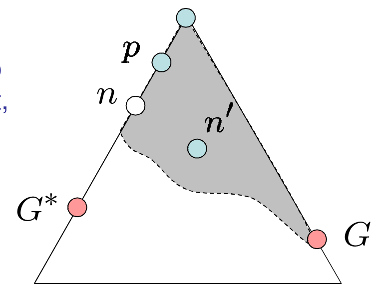
- Very simple fix: never expand a state twice

```
function GRAPH-SEARCH(problem, fringe) returns a solution, or failure
   closed ← an empty set
   fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
   loop do
       if fringe is empty then return failure
       node ← REMOVE-FRONT(fringe)
       if GOAL-TEST(problem, STATE[node]) then return node
       if STATE[node] is not in closed then
           add STATE[node] to closed
           fringe ← INSERTALL(EXPAND(node, problem), fringe)
   end
```

- Can this wreck completeness? Optimality?

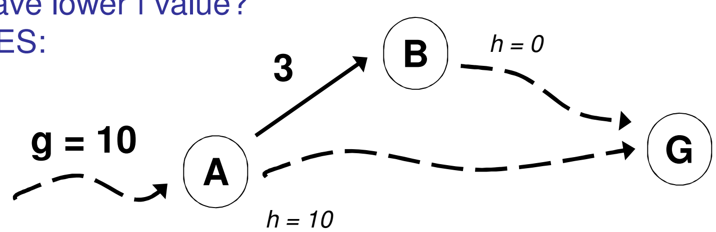
Optimality of A* Graph Search

Proof:
- New possible problem: nodes on path to G* that would have been in queue aren't, because some worse *n'* for the same state as some *n* was dequeued and expanded first (disaster!)
- Take the highest such *n* in tree
- Let *p* be the ancestor which was on the queue when *n'* was expanded
- **Assume *f(p) < f(n)***
- *f(n) < f(n')* because *n'* is suboptimal
- *p* would have been expanded before *n'*
- Contradiction!



Consistency

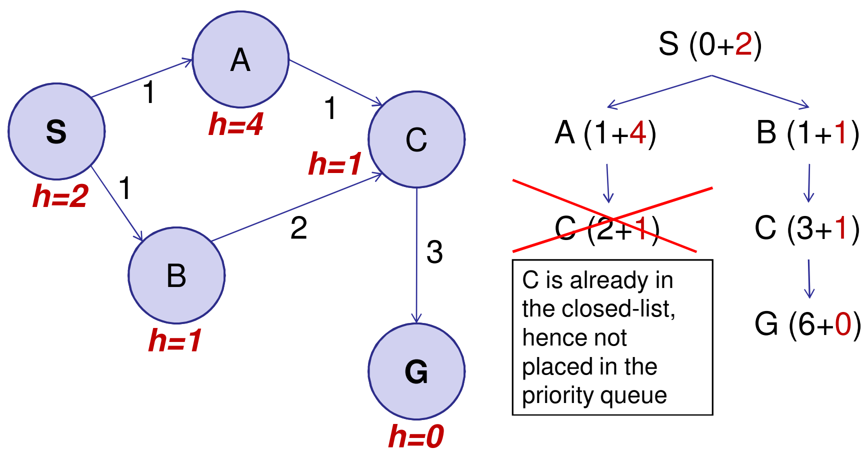
- Wait, how do we know parents have better f-values than their successors?
- Couldn't we pop some node *n*, and find its child *n'* to have lower f value?
- YES:



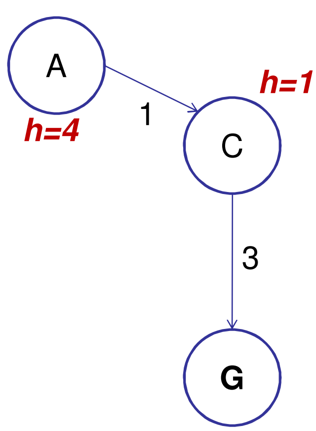
- What can we require to prevent these inversions?
- Consistency: $c(n, a, n') \geq h(n) - h(n')$
- Real cost must always exceed reduction in heuristic

A* Graph Search Gone Wrong

State space graph **Search tree**



Consistency



The story on Consistency:

- Definition:
  cost(A to C) + h(C) ≥ h(A)
- Consequence in search tree:
  Two nodes along a path: $N_{A,}$ N_C
  $g(N_C) = g(N_A) + cost(A\ to\ C)$
  $g(N_C) + h(C) \geq g(N_A) + h(A)$
- The f value along a path never decreases
- Non-decreasing f means you're optimal to every state (not just goals)

Optimality Summary

- Tree search:
  - A* optimal if heuristic is admissible (and non-negative)
  - Uniform Cost Search is a special case (h = 0)
- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)
- Consistency implies admissibility
  - Challenge:Try to prove this.
  - Hint: try to prove the equivalent statement *not admissible implies not consistent*
- In general, natural admissible heuristics tend to be consistent
- Remember, costs are always positive in search!

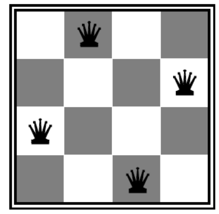
What is Search For?

- Models of the world: single agents, deterministic actions, fully observed state, discrete state space
- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics to guide, fringe to keep backups
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

19

Constraint Satisfaction Problems

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything

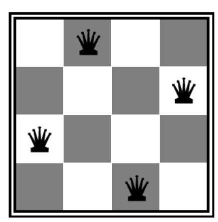


- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms

20

Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0,1\}$
  - Constraints



$$\forall i,j,k \quad (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \quad (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \quad (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \quad (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

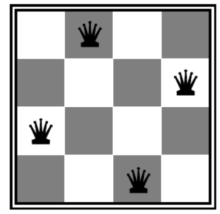
$$\sum_{i,j} X_{ij} = N$$

21

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \dots N\}$
  - Constraints:

Q_1
Q_2
Q_3
Q_4



Implicit: $\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$

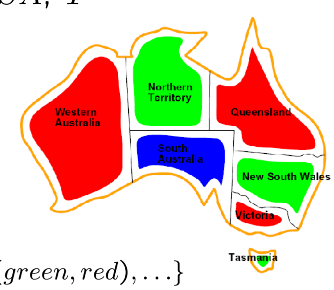
-or-

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \dots\}$

$\dots$

Example: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors



$WA \neq NT$

$(WA, NT) \in \{(red, green), (red, blue), (green, red), \dots\}$

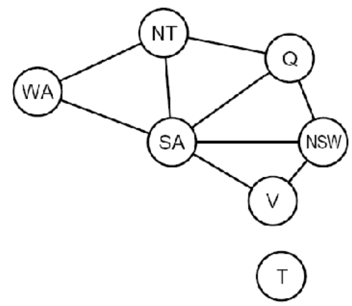
- Solutions are assignments satisfying all constraints, e.g.:

$\{WA = red, NT = green, Q = red,$
$NSW = green, V = red, SA = blue, T = green\}$

24

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



Example: Cryptarithmetic

- Variables (circles):

  $F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$
- Domains:

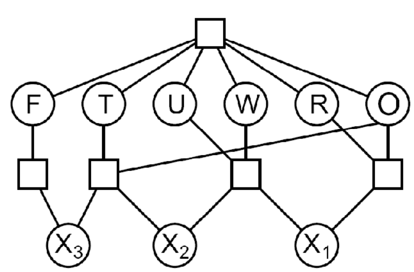
  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints (boxes):

  $\text{alldiff}(F, T, U, W, R, O)$

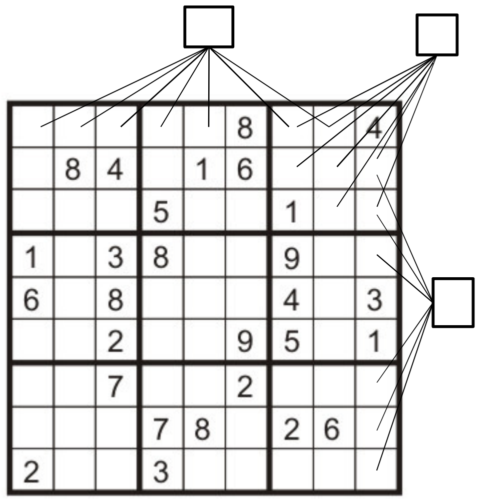
  $O + O = R + 10 \cdot X_1$

  $\ldots$

```
    T W O
+   T W O
---------
  F O U R
```



Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

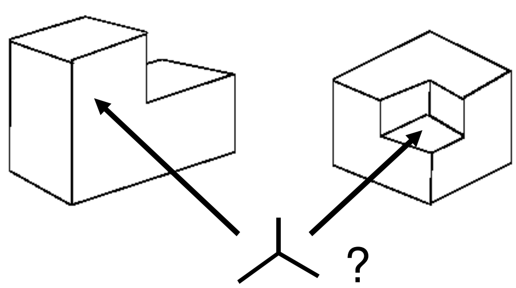
9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra
- An early example of a computation posed as a CSP



- Look at all intersections
- Adjacent intersections impose constraints on each other

28

Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start-end state of a robot
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)

32

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

$$SA \neq green$$

  - Binary constraints involve pairs of variables:

$$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables: e.g., cryptarithmetic column constraints
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

33

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floorplanning
- Fault diagnosis
- … lots more!

- Many real-world problems involve real-valued variables…

34

Standard Search Formulation

- Standard search formulation of CSPs (incremental)

- Let's start with the straightforward, dumb approach, then fix it

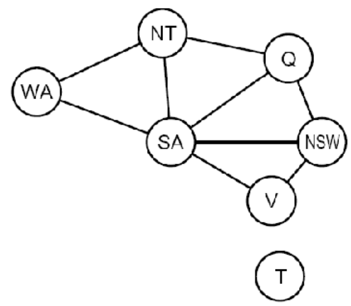
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- Simplest CSP ever: two bits, constrained to be equal

35

Search Methods

- What does BFS do?
- What does DFS do?



- What's the obvious problem here?
- What's the slightly-less-obvious problem?

37

Backtracking Search

- Idea 1: Only consider a single variable at each point
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there?
- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok
  - "Incremental goal test"
- Depth-first search for CSPs with these two improvements is called *backtracking search* (useless name, really)
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for $n \approx 25$

39

Backtracking Search

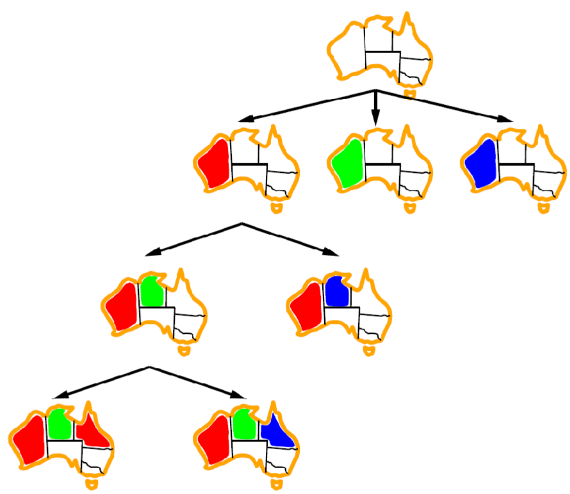
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- What are the choice points?

41

Backtracking Example



42

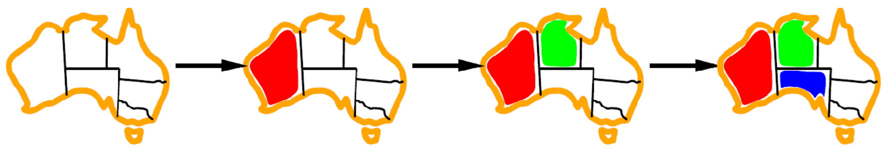
Improving Backtracking

- General-purpose ideas can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

43

Minimum Remaining Values

- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values

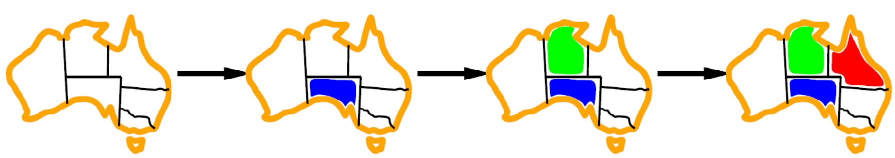


- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering

45

Degree Heuristic

- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable participating in the most constraints on remaining variables

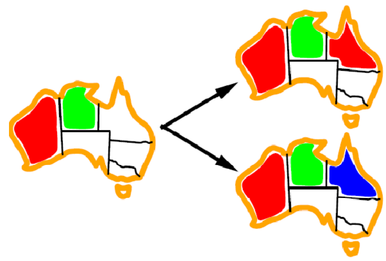


- Why most rather than fewest constraints?

46

Least Constraining Value

- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!
- Why least rather than most?
- Combining these heuristics makes 1000 queens feasible



47

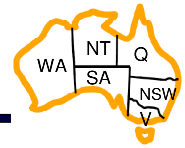
Forward Checking

- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints)
- Idea: Terminate when any variable has no legal values

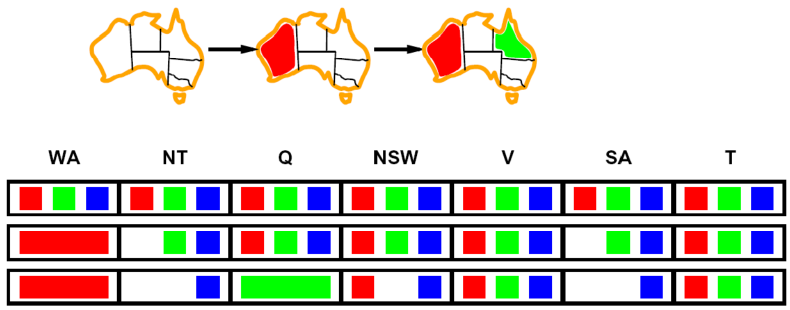
48

[demo: forward checking animation]

Constraint Propagation



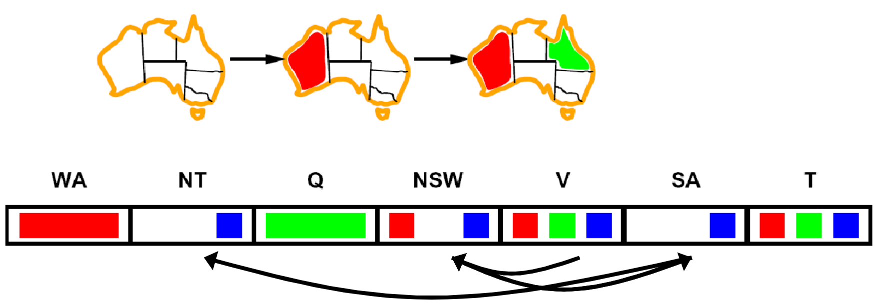
- Forward checking propagates information from assigned to adjacent unassigned variables, but doesn't detect more distant failures:



- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

Arc Consistency

- Simplest form of propagation makes each arc *consistent*
  - X → Y is consistent iff for *every* value x there is *some* allowed y



- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What's the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

Arc Consistency

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

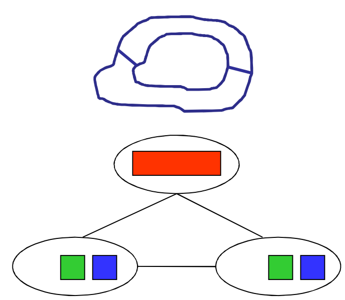
- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

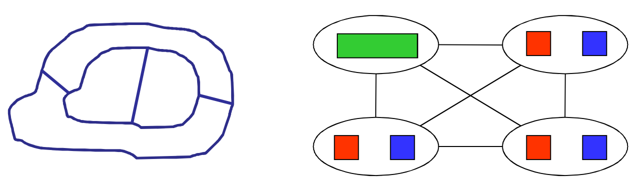
51

[demo: arc consistency animation]

Limitations of Arc Consistency

- After running arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

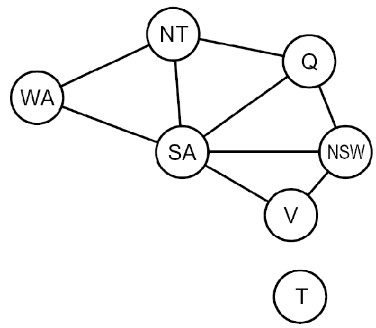




Demo: Backtracking + AC

Problem Structure

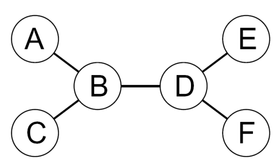
- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec

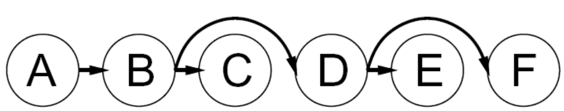


55

Tree-Structured CSPs

- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering

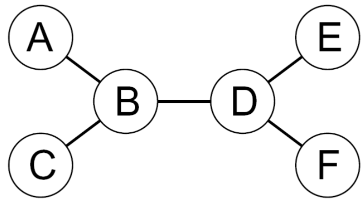




- For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
- For i = 1 : n, assign X_i consistently with Parent(X_i)
- Runtime: $O(n\ d^2)$

56

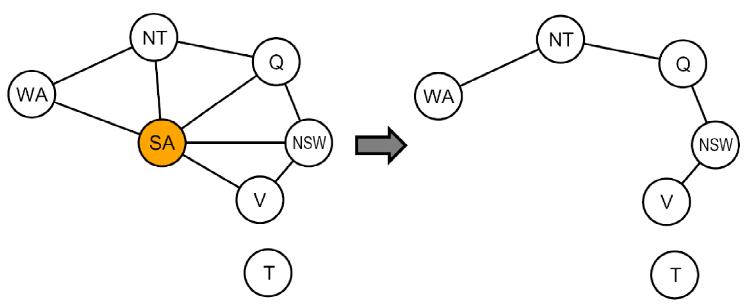
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time!
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

57

Nearly Tree-Structured CSPs



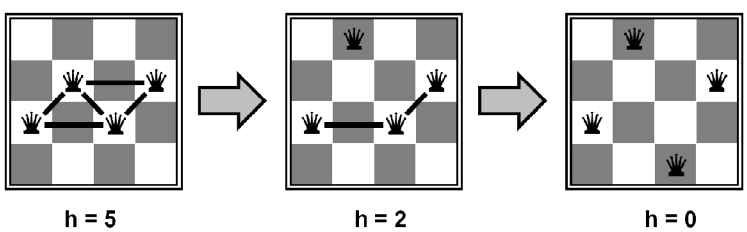
- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O(\ (d^c)\ (n\text{-}c)\ d^2\ )$, very fast for small c

58

Iterative Algorithms for CSPs

- Greedy and local methods typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - Allow states with unsatisfied constraints
  - Operators *reassign* variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens

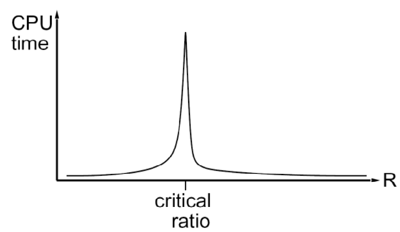


- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search with one legal variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The constraint graph representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice

Local Search Methods

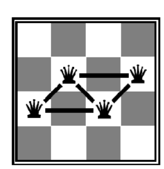
- Queue-based algorithms keep fallback options (backtracking)
- Local search: improve what you have until you can't make it better
- Generally much more efficient (but incomplete)

63

Types of Problems

- Planning problems:
  - We want a path to a solution (examples?)
  - Usually want an optimal path
  - *Incremental formulations*

- Identification problems:
  - We actually just want to know what the goal is (examples?)
  - Usually want an optimal goal
  - *Complete-state formulations*
  - Iterative improvement algorithms



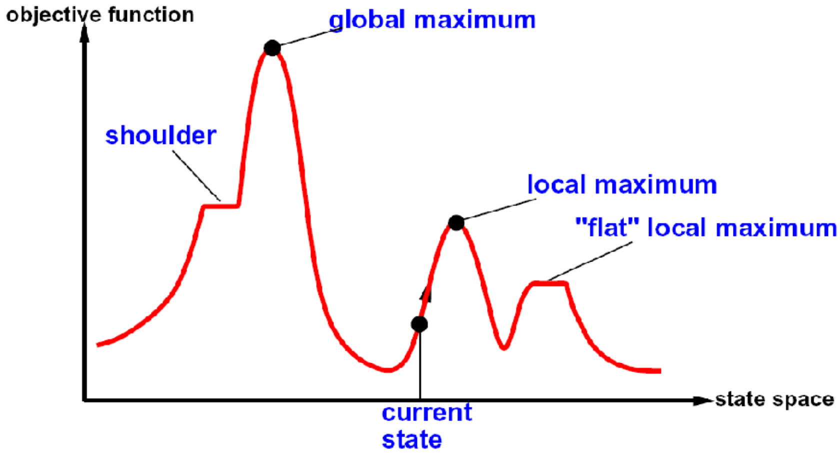
64

Hill Climbing

- Simple, general idea:
  - Start wherever
  - Always choose the best neighbor
  - If no neighbors have better scores than current, quit
- Why can this be a terrible idea?
  - Complete?
  - Optimal?
- What's good about it?

65

Hill Climbing Diagram



- Random restarts?
- Random sideways steps?

66

Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
   inputs: problem, a problem
           schedule, a mapping from time to "temperature"
   local variables: current, a node
                    next, a node
                    T, a "temperature" controlling prob. of downward steps

   current ← MAKE-NODE(INITIAL-STATE[problem])
   for t ← 1 to ∞ do
       T ← schedule[t]
       if T = 0 then return current
       next ← a randomly selected successor of current
       ΔE ← VALUE[next] – VALUE[current]
       if ΔE > 0 then current ← next
       else current ← next only with probability e^(Δ E/T)
```

67

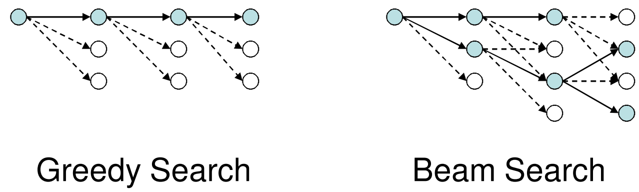
Simulated Annealing

- Theoretical guarantee:
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee?
- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape, the less likely you are to every make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways

68

Beam Search

- Like greedy search, but keep K states at all times:



- Variables: beam size, encourage diversity?
- The best choice in MANY practical settings
- Complete? Optimal?
- What criteria to order nodes by?

69